multiple of 0.07, there differences are also a multiple of 0.07. So $f\left(\frac{49}{5}\right)=f\left(\frac{28}{50}\right)=f(-42)$.

$$f\left(\frac{1}{2}f\left(\frac{28}{50}\right)-\frac{2}{25}f(-42)\right)=f\left(\left(\frac{1}{2}-\frac{2}{25}\right)\frac{11}{3}\right)=f(0.07\cdot 22)=f\left(\frac{49}{5}\right)=\frac{3}{11}$$

Also solved by the proposer.

5623: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let P be an interior point to an equilateral triangle of altitude one. If x, y, z are the distances from P to the sides of the triangle, then prove that

$$x^2+y^2+z^2 \geq x^3+y^3+z^3+6xyz.$$

Solutions 1,2 and 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Solution 1:

Since

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)$$

and

$$x^3+y^3+z^3=(x+y+z)^3-3(x+y+z)(xy+yz+zx)+3xyz,$$

the inequality to prove is equivalent to

$$(x+y+z)^2-2(xy+yz+zx)^3 \geq (x+y+z)^3-3(x+y+z)(xy+yz+zx)+9xyz.$$

From Viviani's theorem, $x+y+z=1$, so this last inequality is the same as

$$xy+yz+zx \geq 9xyz,$$

that is (since x, y, z are strictly positive because P is interior to the equilateral triangle)

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 9.$$

or, since $x+y+z=1$, to

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3+\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}=3+\frac{x}{y}+\frac{1}{\frac{x}{y}}+\frac{y}{z}+\frac{1}{\frac{y}{z}}+\frac{z}{x}+\frac{1}{\frac{z}{x}} \geq$$

$$\geq 3+2+2+2=9,$$

where we have used that $t+\frac{1}{t} \geq 2$ for any $t>0$, with equality if and only if $t=1$ (which is equivalent to $t^2-2t+1 \geq 0$, that is, to $(t-1)^2 \geq 0$) with $t=\frac{x}{y}, t=\frac{y}{z}$ and $t=\frac{z}{x}$.

Thus, the inequality is proven and also that equality occurs if and only if, $\frac{x}{y} = \frac{y}{z} = \frac{z}{x} = 1$, that is, $x = y = z = \frac{1}{3}$. In plane geometry the barycenter of a triangleis is also called the centroid of the triangle. Is the point in the triangle where the three medians meet. In equilateral triangles, the incenter, circumcenter and orthocenter also meet at the barycenter.

Solution 2:

From Viviani's theorem, $x + y + z = 1$, so the equality in the problem is successively equivalent to

$$x^2 - x^3 + y^2 - y^3 + z^2 - z^3 \geq 6xyz,$$

$$x^(1 - x) + y^2(1 - y) + z^2(1 - z) \geq 6xyz,$$

$$x^2(y + z) + y^2(z + x) + z^2(x + y) \geq 6xyz,$$

$$xy(x + y) + yz(y + z) + zx(z + x) \geq 6xyz.$$

Since P is interior to the triangle, $x > 0, y > 0$ and $z > 0$, so this last inequality (and, hence, the inequality in the problem) is equivalent to

$$\frac{z}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z} \geq 6,$$

which is true because, for any $t > 0$, we have $t + \frac{1}{t} \geq 2$ with equality if and only if $t = 1$, so taking $t = x/y$, $t = y/z$, and $t = z/x$ and adding the obtained inequalities we obtain the required result, and note that equality occurs on it if, and only if, $x/y = y/z = z/x = 1$; that is, if and only if $x = y = z = 1/3$. This means that P is the barycenter of the equilateral triangle (see definition above).

Solution 3:

From Viviani's theorem $x + y + z = 1$, and we can think of this as a constraint and apply the Lagrange multiplers method to the funion $L(x, y, z; \lambda) = f(x, y, z) - \lambda g(x, y, z)$ where $f(x, y, z) = x^2 + y^2 + z^2 - x^3 - y^3 - z^3 - 6xyz$ is the objective function that is subjected to the constraint constraint $g(x, y, z) = x + y + z - 1$. The critical points in this expression are points $(x, y, z) \in (0, +\infty)$ for which

$$\frac{\partial L}{\partial x}(x, y, z) = \frac{\partial L}{\partial y}(x, y, z) = \frac{\partial L}{\partial z}(x, y, z) = \frac{\partial L}{\partial \lambda}(x, y, z) = 0.$$

Thus, $2x - 3x^2 - 6yz = 2y - 3y^2 - 6zx = 2z - 3z^2 - 6xy = \lambda$ and $x + y + z = 1$, That is

$0 = 2(x - y) - 3(x^2 - y^2) - 6z(x - y) = (2 - 3x = 3y = 6z)(x - y)$ and

$0 = 2(u - z) - 3(y^2 - z^2) - 6x(y - z) = (2 - 3y - 3z - 6x)(y - z)$ and $x + y + z = 1$. From this it follows that:

$(x,y,z)\in\left\{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\ \left(\frac{1}{9},\frac{1}{9},\frac{7}{9}\right)\right\},\left(\frac{1}{9},\frac{7}{9},\frac{1}{9}\right),\left(\frac{7}{9},\frac{1}{9},\frac{1}{9}\right)\Big\}$ with $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ being the point where f attains its minimum value.

In summary,:

$f(x,y,z)\geq(1/3,1/3,1/3)=0$ for any $(x,y,z)\in(0,+\infty)$; equality is obtained if, and only if, $(x,y,z)=(1/3,1/3,1/3)$.

Solution 4 by Michael Brozinsky, Central Islip,NY

We first note that if $A+B=C+D$ then $A\geq C$ is equivalent to $B\leq D$. Without loss of generality let $x\leq y\leq z$ and let $k=x+y$ so that $k\leq\dfrac{2}{3}$ since $x+y+z=1$ as the sum of the distances from an interior of point of an equilateral triangle is easily shown (by dissection) to be equal to the altitude. Now since

$$\begin{aligned}x^2+y^2+z^2+2xy+2xz+2yz &= (x+y+z)^2=1=(x+y+z)^3=\\ &= x^3+y^3+z^3+6xyz+3x^2y+3xy^2+3xz^2+3y^2z+3yz^2\end{aligned}$$

it suffices to show (by our first note above) that

$$2xy+2xz_+2yz\leq 3x^2y+3xy^2+3x^2+3x^2z+3xz^2+3y^2z+3yz^2$$

or replacing z by$1-k$ and y by $k-x$

$$\begin{aligned}&2x(k-x)+2x(1-k)+2(k-x)(1-k)-(3x^2(k-x)+3x(k-x)^2+\\ +\ &3x^2(1-k)+3x(1-k)^2+3(k-x)^2(1-k)+3(k-x)(1-k)^2\leq 0\end{aligned}$$

which simplifies to

$$(9k-8)x^2+(-3k^2+6k(1-k)+2k)x-3k^2(1-k)-3k(1-k)^2+2k(1-k)\leq 0. \qquad (*)$$

Now the left hand side of $(*)$ viewed as a quadratic in x has a negative leading coefficient since $k\leq\dfrac{2}{3}<\dfrac{8}{9}$, and by completing the square becomes

$$(9k-8)\left(x-\frac{1}{2}k\right)^2-\frac{1}{4}\left(-2+3k\right)k$$

which is thus less than or equal to 0 with equality only if

$$k-\frac{2}{3}\ \textbf{and}\ x=\frac{1}{2}k=\frac{1}{3}\ \textbf{and thus}\ y=\frac{1}{3}\ \textbf{and}\ z=\frac{1}{3}.$$

Solution 5 by Titu Zvonaru, Comănesti, Romania

Let a be the side of the equilateral triangle and $h = 1$ its altitude. It is easy to see that $ax + ay + az = ah$, hence $x + y + z = 1$. The desired inequality is equivalent to

$$(x + y + z)(x^2 + y^2 + z^2) \geq x^3 + y^3 + z^3 + 6xyz$$

$$x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2 \geq 6xyz,$$

which is true by the AM-GM inequality for positive numbers $x^2y, xy^2, y^2z, yz^2, x^2z, xz^2$.

Solution 6 by (Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By Viviani's theorem, $x + y + z = 1$. By multiplying the left-hand side by $x + y + z$, the inequality becomes homogeneous. Then, after clarifying the inequality becomes

$$x^2y + x^2z + xy^2 + yz^2 + y^2z + xz^2 \geq 6xyz$$

which follows by the AM-GM inequality.

Solution 7 by Arkady Alt, San Jose, California

Let F and a be, respectively, the area and sidelength of the triangle.

Then $F = \dfrac{x \cdot a}{2} + \dfrac{y \cdot a}{2} + \dfrac{z \cdot a}{2} = \dfrac{a \cdot 1}{2}$.Hence, $x+y+z = 1$ and by AM-GM Inequality we have

$x^2 + y^2 + z^2 = \left(x^2 + y^2 + z^2\right)(x + y + z) =$

$x^3 + y^3 + z^3 + \sum_{cyc} x^2 (y + z) \geq x^3 + y^3 + z^3 + \sum_{cyc} x^2 \cdot 2\sqrt{yz} \geq$

$x^3 + y^3 + z^3 + 6\sqrt[3]{x^2y^3z^3} = x^3 + y^3 + z^3 + 6xyz.$

Solution 8 by Albert Natian, Los Angeles Valley College, Valley Glen, California/

Lemma. Suppose $x, y, z \in [0, 1]$ and define

$$s := x + y + z, \quad \sigma := x^2 + y^2 + z^2, \quad w := x^3 + y^3 + z^3, \quad p := xyz.$$

If $s = 1$, then $\sigma \geq w + 6p$.

Proof. Suppose $s = 1$. We will show via the method of Lagrange Multipliers that

$$Q := \sigma - w - 6p \geq 0.$$

But first we make the observation that

$$1 = s^3 = (x + y + z)^3 = -2w + 3\sigma + 6p,$$

$$6p = 1 + 2w - 3\sigma.$$

So

$$Q = \sigma - w - 6p = \sigma - w - (1 + 2w - 3\sigma),$$

$$Q = 4\sigma - 3w - 1.$$

The Lagrangian $\mathcal{L}$, with multiplier λ, is given by

$$\mathcal{L} = 4\sigma - 3w - 1 - \lambda s$$

which yields

$$\frac{\partial \mathcal{L}}{\partial x} = 8x - 9x^2 - \lambda, \quad \frac{\partial \mathcal{L}}{\partial y} = 8y - 9y^2 - \lambda, \quad \frac{\partial \mathcal{L}}{\partial z} = 8z - 9z^2 - \lambda.$$

Setting the latter three partials equal to zero, followed by re-arrangement, we get

$$9x^2 - 8x + \lambda = 0, \quad 9y^2 - 8y + \lambda = 0, \quad 9z^2 - 8z + \lambda = 0,$$

each of which has solution

$$\frac{1}{9}\left(4 \pm \sqrt{16 - 9\lambda}\right)$$

resulting in either $\lambda = 5/3$ or $\lambda = 7/9$. Only for $\lambda = 5/3$ is Q minimized with minimum value 0 at $x = y = z = 1/3$. Thus $\sigma \geq w + 6p$.

If P were an interior point to any triangle of side lengths a, b and c, with x, y, z being, respectively, the distances from P to the sides with lengths a, b and c, then the area $\mathcal{A}$ of the triangle can be expressed as

$$\mathcal{A} = \frac{1}{2}\left(ax + by + cz\right).$$

For an equilateral triangle with side length a,

$$\mathcal{A} = \frac{1}{2}a\left(x + y + z\right).$$

The side a and area $\mathcal{A}$ of an equilateral triangle whose altitude is one are given by $a = 2/\sqrt{3}$ and $\mathcal{A} = 1/\sqrt{3}$. Thus

$$\frac{1}{\sqrt{3}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}}\left(x + y + z\right),$$
$$x + y + z = 1$$

which allows us to conclude, by the above Lemma, that

$$\sigma \geq w + 6p,$$

$$x^2 + y^2 + z^2 \geq x^3 + y^3 + z^3 + 6xyz.$$

Solution 9 by Samuel Aguilar (student) and the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA

By Viviani's Theorem, $x + y + z = 1$. (See, e.g., Ken-Ichiroh Kawasaki, "Proof Without Words: Viviani's Theorem," *Math. Mag.* **78**(3), 2005, p. 213.)

By the AM-HM Inequality,

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x + y + z}{3} = \frac{1}{3},$$

and

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 9.$$

Thus, multiplying both sides by xyz and subtracting $3xyz$, we get

$$\begin{aligned} yz+zx+xy-3xyz &\geq 6xyz \\ yz(1-x)+zx(1-y)+xy(1-z) &\geq 6xyz \\ x^3+y^3+z^3+yz(y+z)+zx(z+x)+xy(x+y) &\geq x^3+y^3+z^3+6xyz \\ (x+y+z)(x^2+y^2+z^2) &\geq x^3+y^3+z^3+6xyz, \end{aligned}$$

which gives the desired inequality, since $x+y+z=1$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, Westchester Area Math Circle Purchase, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5624: *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia*

Evaluate: $\displaystyle\int_0^1\left(\frac{\tan^{-1}x-x}{x^2}\right)^2dx$

Solution 1 by Moti Levy, Rehovot, Israel

$$I:=\int_0^1\left(\frac{\arctan(x)-x}{x^2}\right)^2dx=\int_0^1\frac{1}{x^4}\left(\arctan(x)-x\right)^2dx.$$

By integration by parts,

$$\begin{aligned} I &= -\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2+\frac{2}{3}\int_0^1\frac{1}{x\left(x^2+1\right)}\left(x-\arctan(x)\right)dx \\ &= -\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2+\frac{2}{3}\int_0^1\frac{1}{\left(x^2+1\right)}dx-\frac{2}{3}\int_0^1\frac{\arctan(x)}{x\left(x^2+1\right)}dx \\ &= -\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2+\frac{\pi}{6}-\frac{2}{3}\int_0^1\frac{\arctan(x)}{x\left(x^2+1\right)}dx. \end{aligned}$$

Using the partial fractions of $\frac{1}{x(x^2+1)}=\frac{1}{2x}+\frac{1-x^2}{2x(1+x^2)}$, we get

$$\int_0^1\frac{\arctan x}{x\left(x^2+1\right)}dx=\frac{1}{2}\int_0^1\frac{\arctan x}{x}dx+\int_0^1\arctan(x)\frac{1-x^2}{2x\left(1+x^2\right)}dx.$$

The first integral is related to Catalan constant G

$$G=\int_0^1\frac{\arctan x}{x}dx$$